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1990 J. Phys. A: Math. Gen. 23 2027

(http://iopscience.iop.org/0305-4470/23/11/026)

J. Phys. A: Math. Gen. 23 (1990) 2027-2047. Printed in the UK

Quasiclassical states of the Coulomb system and so(4, 2)

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Received 6 December 1989, in final form 6 February 1990

Abstract. Quasiclassical bound states of the quantum mechanical Coulomb system are constructed. They are initially Barut-Girardello coherent states of the natural so(4, 2) dynamical algebra, and evolve in accordance with the Schrödinger equation. The asymptotic behaviour of expectation values and uncertainties of all the so(4, 2) observables is examined in detail for those states having a mean value of n, the principal quantum number, approaching infinity. There is no spreading with respect to these observables over times of the order of τ, the corresponding classical period, but the states do spread over times of the order of $\tau^{7/6}$. Periodic but successively weaker resurgences of coherence are found over times of the order of $\tau^{4/3}$. It is shown explicitly that the states are quasiperiodic over extremely long times.

1. Introduction

Schrödinger (1926) constructed special wavefunctions $\psi(x, t)$ for the simple harmonic oscillator in quantum mechanics, corresponding to states which today we call coherent states. For any such state, the probability density $\psi^*(x, t)\psi(x, t)$ does not spread in the coordinate x with increasing time t, and remains localised near $\hat{x}(t)$, a solution of the classical equations of motion defining a trajectory of the classical oscillator. In these states the system therefore behaves to a good approximation like a classical particle performing simple harmonic motion.

In the same paper, Schrödinger raised the possible existence of analogous 'quasi-classical states' for the Coulomb system. It is understood now that, because of the nonlinear dynamics in the Coulomb case, genuinely 'non-spreading' states that follow a classical trajectory do not exist. However, there can exist states for which the rate of spreading, measured in terms of the expectation values of suitably chosen dynamical variables, is slow in a well defined sense, and such states may also be called quasi-classical. There have been several attempts to construct states of this type for the Coulomb system.

Brown (1973) superposed bound energy eigenstates in order to construct quasi-classical states corresponding to circular orbits. These states spread over times of the order $\tau^{7/6}$, but not over times of the order τ, where τ is the corresponding classical period. In this respect they are qualitatively similar to the states we shall construct below, but as well as being limited to circular orbits, Brown's method can be criticised for its somewhat *ad hoc* nature. Mostowski (1977) defined (bound) coherent states for the SO(4, 2) dynamical group of the Coulomb system, following Perelomov's (1972) general procedure, and allowed these states to evolve under the action of the Coulomb Hamiltonian. He stated that the resultant states behave quasiclassically, with significant spreading only over times of order $\tau^{7/6}$, the spreading being measured by the uncertainties of observables corresponding to the group generators. However the details of his calculations do not seem to have been published, and our own calculations do not support his conclusions. (See the comments on Perelomov coherent states in section 3.) Nieto and Simmons (1979) and Gutschick and Nieto (1979) used their method of 'minimum uncertainty coherent states' to define quasiclassical bound states for the radial operators of the Coulomb system, but did not consider the full three-dimensional problem, nor determine a characteristic time for spreading. Bhaumik *et al* (1986) used the Kustaanheimo-Stiefel (1965) transformation to convert the classical Coulomb problem into a constrained four-dimensional isotropic harmonic oscillator problem, and then used coherent states of the corresponding quantum oscillator to define quasiclassical bound states for the quantum Coulomb system. They showed that these states also spread only over times of the order $\tau^{7/6}$, and gave a detailed discussion of behaviour in the case of circular orbits. However, in the case of elliptic orbits, they stated without proof that spreading occurs over a circular annulus (presumably with its centre at the centre of attraction), a result that appears inconsistent with the constancy of the Runge-Lenz vector (and of its uncertainty). Gerry (1986) and Gerry and Kiefer (1988) constructed quasiclassical bound states which depend on 'fictitious' time variables; these states do not evolve in accordance with the Schrödinger equation. The quasiclassical states constructed by Garbaczewski and Prorok (1987) also fail to satisfy that equation.

Our object in what follows is to describe quasiclassical bound states based on an so(4, 2) dynamical algebra. Initially, such states are taken to be so(4, 2) coherent states in the sense of Barut and Girardello (1971), and they then evolve in accordance with the Schrödinger equation. For a given classical orbit, either circular or elliptical, these states do not spread over times of the order τ, the corresponding classical period, but only over times of the order $\tau^{7/6}$. Spreading occurs around an orbit but not away from it, in a manner that is, in particular, necessarily consistent with the constancy of the Runge-Lenz vector. For these states, spreading is defined in terms of the uncertainties of the observables corresponding to the so(4, 2) generators, which are all Hermitian.

2. The dynamical algebra so(4, 2)

It is well known (Malkin and Man'ko 1965, Bacry 1966, Musto 1966, Pratt and Jordan 1966, Barut and Kleinert 1967a, b, Fronsdal 1967, Nambu 1967, Györgyi 1968, 1969, Barut 1972; Englefield 1972) that SO(4, 2) (so(4, 2)) is a dynamical group (algebra) for the Coulomb system. In earlier work (McAnally and Bracken 1988), the following explicit expressions have been derived for the so(4, 2) generators, acting in the subspace of bound states of the system;

$$\Gamma_0(=N)=[-2H]^{-1/2} \qquad \boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p} \qquad \boldsymbol{A}=\tfrac{1}{2}(\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{L}\times\boldsymbol{p}-2\boldsymbol{r}r^{-1})N$$
$$\Gamma_4\pm \mathrm{i}T=\tfrac{1}{2}\{rp^2U_\pm[N\bullet 1]-rU_\pm[N\pm 1]^{-1}\bullet 2\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{p}-\mathrm{i})U_\pm\}N[N\bullet 1]^{-1} \tag{1}$$
$$\boldsymbol{\Gamma}\pm \mathrm{i}\boldsymbol{M}=\{r\boldsymbol{p}U_\pm\pm\mathrm{i}[\tfrac{1}{2}\boldsymbol{r}p^2-(\boldsymbol{r}\cdot\boldsymbol{p}-\mathrm{i})\boldsymbol{p}]U_\pm[N\pm 1]\pm\tfrac{1}{2}\mathrm{i}\boldsymbol{r}U_\pm[N\pm 1]^{-1}\}N[N\pm 1]^{-1}.$$

Here N is the number operator, whose eigenvalue $n\in\{1,2,\ldots\}$ is the principal quantum number of the Coulomb system. On the ground state ($n=1$), where their definitions in (1) break down, $\Gamma_4-\mathrm{i}T$ and $\boldsymbol{\Gamma}-\mathrm{i}\boldsymbol{M}$ are taken to vanish. We have set to unity the (reduced) mass m, Planck's constant $\hbar$, and the coefficient Ze^2 in the potential, so that the Hamiltonian has the form

$$H = \tfrac{1}{2}p^2 - 1/r \tag{2}$$

with $r = [\boldsymbol{r}\cdot\boldsymbol{r}]^{1/2}$, $p = [\boldsymbol{p}\cdot\boldsymbol{p}]^{1/2}$. In addition,

$$U_\pm = \exp\{\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{p} - \mathrm{i}) : \ln(N[N \bullet 1]^{-1})\} \tag{3}$$

where the 'ordered exponential' is defined by

$$\exp\{A : B\} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n B^n \tag{4}$$

and multiples of the identity operator have been represented throughout by the corresponding complex numbers. (Note that in our earlier paper, we denoted Γ_0 by Γ_0^*, $\boldsymbol{A}$ by $\boldsymbol{A}^*$, etc.) The Hermiticity of the operators (1) (with respect to the usual scalar product, for which $\boldsymbol{r}$ and $\boldsymbol{p}$ are Hermitian) can be checked by evaluating their matrix elements between bound states in the coordinate representation. In order to check that they satisfy the so(4, 2) commutation relations, we note that (McAnally and Bracken 1988)

$$\Gamma_0 = K^{-1}\tilde{\Gamma}_0 K \qquad \boldsymbol{A} = K^{-1}\tilde{\boldsymbol{A}}K \qquad \text{etc} \tag{5}$$

where (Barut 1972)

$$\begin{aligned}
&\tilde{\Gamma}_0 = \tfrac{1}{2}(rp^2 + r) \qquad \tilde{\Gamma}_4 = \tfrac{1}{2}(rp^2 - r) \qquad \tilde{T} = \boldsymbol{r}\cdot\boldsymbol{p} - \mathrm{i} \\
&\tilde{\boldsymbol{L}} = \boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p} \qquad \tilde{\boldsymbol{\Gamma}} = r\boldsymbol{p} \\
&\tilde{\boldsymbol{A}} = \tfrac{1}{2}\boldsymbol{r}p^2 - (\boldsymbol{r}\cdot\boldsymbol{p} - \mathrm{i})\boldsymbol{p} - \tfrac{1}{2}\boldsymbol{r} \\
&\tilde{\boldsymbol{M}} = \tfrac{1}{2}\boldsymbol{r}p^2 - (\boldsymbol{r}\cdot\boldsymbol{p} - \mathrm{i})\boldsymbol{p} + \tfrac{1}{2}\boldsymbol{r}
\end{aligned} \tag{6}$$

and K is the 'tilt' transformation

$$\begin{aligned}
K &= \exp\{\mathrm{i}\tilde{T} : \ln N\}N \\
K^{-1} &= \exp\{-\mathrm{i}\tilde{T} : \ln \tilde{\Gamma}_0\}\tilde{\Gamma}_0^{-1}.
\end{aligned}$$

The so(4, 2) commutation relations between the operators (6) are easily checked (Barut 1972) and it follows from (5) that the operators (1) also satisfy such relations (McAnally and Bracken 1988), namely

$$\begin{aligned}
&[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k \qquad [L_i, M_j] = \mathrm{i}\epsilon_{ijk}M_k \\
&[L_i, A_j] = \mathrm{i}\epsilon_{ijk}A_k \qquad [L_i, \Gamma_j] = \mathrm{i}\epsilon_{ijk}\Gamma_k \\
&[\Gamma_0, \Gamma_4] = \mathrm{i}T \qquad [\Gamma_4, T] = -\mathrm{i}\Gamma_0 \qquad [T, \Gamma_0] = \mathrm{i}\Gamma_4 \\
&[\Gamma_0, \boldsymbol{M}] = -\mathrm{i}\boldsymbol{\Gamma} \qquad [\Gamma_0, \boldsymbol{\Gamma}] = \mathrm{i}\boldsymbol{M} \qquad [\Gamma_4, \boldsymbol{A}] = \mathrm{i}\boldsymbol{\Gamma} \\
&[\Gamma_4, \boldsymbol{\Gamma}] = \mathrm{i}\boldsymbol{A} \qquad [T, \boldsymbol{A}] = \mathrm{i}\boldsymbol{M} \qquad [T, \boldsymbol{M}] = \mathrm{i}\boldsymbol{A} \\
&[A_i, A_j] = \mathrm{i}\epsilon_{ijk}L_k \qquad [M_i, M_j] = -\mathrm{i}\epsilon_{ijk}L_k \qquad [\Gamma_i, \Gamma_j] = -\mathrm{i}\epsilon_{ijk}L_k \\
&[A_i, M_j] = \mathrm{i}T\delta_{ij} \qquad [A_i, \Gamma_j] = \mathrm{i}\Gamma_4\delta_{ij} \qquad [M_i, \Gamma_j] = \mathrm{i}\Gamma_0\delta_{ij}
\end{aligned} \tag{7}$$

with all other commutators vanishing. If we put

$$\begin{aligned}
&J_{ij} = \epsilon_{ijk}L_k \qquad J_{i4} = A_i \qquad J_{i5} = M_i \qquad J_{i6} = \Gamma_i \qquad i, j = 1, 2, 3 \\
&J_{45} = T \qquad J_{46} = \Gamma_4 \qquad J_{56} = \Gamma_0
\end{aligned}$$

then the relations (7) are equivalent to

$$[J_{AB}, J_{CD}] = \mathrm{i}(g_{AC}J_{BD} + g_{BD}J_{AC} - g_{AD}J_{BC} - g_{BC}J_{AD})$$

where $g = \mathrm{diag}(1, 1, 1, 1, -1, -1)$, so that the algebra is indeed isomorphic to so(4, 2).

Despite their complicated forms as functions of $\boldsymbol{r}$ and $\boldsymbol{p}$, the operators (1) are natural dynamical variables for the (bound states of the) Coulomb system. In particular, Γ_0, $\boldsymbol{L}$, $\boldsymbol{A}$ are constants of the motion and, in the Heisenberg picture, $\Gamma_4 \pm \mathrm{i}T$, $\boldsymbol{\Gamma} \pm \mathrm{i}\boldsymbol{M}$ have a simple time-dependence (McAnally and Bracken 1988):

$$\begin{aligned}\Gamma_4(t) \pm \mathrm{i}T(t) &= (\Gamma_4(0) \pm \mathrm{i}T(0)) \exp(\pm\mathrm{i}[N \pm \tfrac{1}{2}]N^{-2}[N \pm 1]^{-2}t)\\ \boldsymbol{\Gamma}(t) \pm \mathrm{i}\boldsymbol{M}(t) &= (\boldsymbol{\Gamma}(0) \pm \mathrm{i}\boldsymbol{M}(0)) \exp(\pm\mathrm{i}[N \pm \tfrac{1}{2}]N^{-2}[N \pm 1]^{-2}t).\end{aligned} \tag{8}$$

(Note however that in the present paper we work in the Schrödinger picture). The classical analogues of the variables (1) have simple meanings in terms of the geometry of an orbit (McAnally and Bracken 1988). This is well known for the constants of the motion Γ_0, $\boldsymbol{L}$, $\boldsymbol{A}$. Figure 1 shows the meaning of the remaining classical variables Γ_4, T, $\boldsymbol{\Gamma}$ and $\boldsymbol{M}$: in the case of the scalar variables, Γ_4 is given by εOX and T by εOY, where ε is the eccentricity of the orbit,

$$\varepsilon = \sqrt{1 + 2HL^2}.$$

As the classical limit of (8) shows (McAnally and Bracken 1988), these remaining quantities all vary periodically in time with the classical period $\tau = 2\pi[-2E]^{-3/2}$, where E is the energy.

It should be stressed that in the calculations that follow, the formulae (1) for the so(4, 2) variables in terms of $\boldsymbol{r}$ and $\boldsymbol{p}$ are largely irrelevant. What are important are the simple so(4, 2) commutation relations (7), the particular irreducible representation of so(4, 2) involved, and the relationship between the so(4, 2) algebra and the dynamics of the system, as determined by the relation $\Gamma_0 = [-2H]^{-1/2}$.

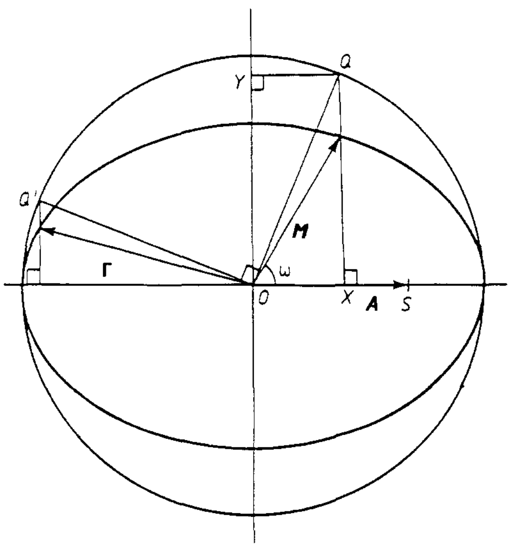

Figure 1. An elliptical orbit in the xy plane rescaled to have semimajor axis Γ_0, showing $\boldsymbol{A}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{M}$; Γ_4/ϵ and T/ϵ equal the Cartesian coordinates X, Y of the point Q. The lines OQ and OQ′ are perpendicular. Angle ω is the 'mean anomaly', and S corresponds to the centre of attraction.

We recall (Mack and Todorov 1969, Barut 1972) that the relevant representation of so(4, 2) here is one from the 'ladder' series of degenerate representations. In particular, Γ_0 has eigenvalues $n = 1, 2, 3, \ldots$ as already noted, and $\Gamma_4 + \mathrm{i}T$, $\boldsymbol{\Gamma} + \mathrm{i}\boldsymbol{M}$ ($\Gamma_4 - \mathrm{i}T$, $\boldsymbol{\Gamma} - \mathrm{i}\boldsymbol{M}$) are raising (lowering) operators for that eigenvalue. Certain 'representation relations' hold (Barut and Böhm 1970), in particular,

$$
\begin{aligned}
&(\Gamma_4)^2 + T^2 = (\Gamma_0)^2 - \boldsymbol{L}^2 = \boldsymbol{A}^2 + 1 \\
&(\Gamma_0)^2 - T^2 = \boldsymbol{M}^2 - 1 \qquad (\Gamma_0)^2 - (\Gamma_4)^2 = \boldsymbol{\Gamma}^2 - 1 \\
&\boldsymbol{\Gamma}^2 + \boldsymbol{M}^2 = (\Gamma_0)^2 + \boldsymbol{L}^2 + 2 \\
&(\Gamma_1)^2 + (M_1)^2 = \boldsymbol{L}^2 - (L_1)^2 + (A_1)^2 + 1 \\
&(\Gamma_2)^2 + (M_2)^2 = \boldsymbol{L}^2 - (L_2)^2 + (A_2)^2 + 1 \\
&(\Gamma_3)^2 + (M_3)^2 = \boldsymbol{L}^2 - (L_3)^2 + (A_3)^2 + 1 \\
&\boldsymbol{L} \cdot \boldsymbol{A} = \boldsymbol{L} \cdot \boldsymbol{M} = \boldsymbol{L} \cdot \boldsymbol{\Gamma} = 0 \\
&\boldsymbol{\Gamma}^2 - \boldsymbol{M}^2 + (\Gamma_4)^2 - T^2 = 0 \\
&\boldsymbol{M} \cdot \boldsymbol{\Gamma} + \boldsymbol{\Gamma} \cdot \boldsymbol{M} + T\Gamma_4 + \Gamma_4 T = 0.
\end{aligned}
\tag{9}
$$

These relations imply limitations on the way in which the expectation values and uncertainties of the variables (1) can vary with time.

Having to hand the so(4, 2) dynamical algebra, we can now proceed in either of two ways to try and construct quasiclassical bound states for the quantum Coulomb system. Barut–Girardello (1971) coherent states for the algebra so(4, 2) can be constructed as common eigenstates of the commuting lowering operators $\Gamma_4 - \mathrm{i}T$, $\boldsymbol{\Gamma} - \mathrm{i}\boldsymbol{M}$; or Perelomov (1972) coherent states for the group SO(4, 2) can be constructed by allowing finite group element representatives to act on the ground state. In either case, such coherent states can then be taken as initial states of the system and allowed to evolve under the action of the Schrödinger equation with the Hamiltonian (2), i.e. $H = -\frac{1}{2}[\Gamma_0]^{-2}$. The idea behind each approach is that, because of the simple commutation relations between the so(4, 2) generators and Γ_0 (and hence H), the inevitable spreading of the states with time, or 'loss of coherence', may be minimal. The approach based on Perelomov coherent states is essentially equivalent to that taken by Mostowski (1977) but, as we see in the next section, this approach does not lead to states which can sensibly be called quasiclassical.

3. The subalgebra so(2, 1) and the one-dimensional Coulomb problem

To illustrate the construction of quasiclassical states in a simplified framework, we imagine a system with an so(2, 1) dynamical algebra generated by Hermitian operators $\Gamma_0(=N)$, Γ_4, T, with

$$[\Gamma_0, \Gamma_4] = \mathrm{i}T \qquad [\Gamma_4, T] = -\mathrm{i}\Gamma_0 \qquad [T, \Gamma_0] = \mathrm{i}\Gamma_4. \tag{10}$$

The relevant representation of so(2, 1) is one from the ladder series with lowest weight 1 (Barut and Fronsdal 1965), and is spanned by an orthonormal set of vectors $|n\rangle$, $n = 1, 2, \ldots$ satisfying

$$\Gamma_0|n\rangle = n|n\rangle \qquad (\Gamma_4 \pm \mathrm{i}T)|n\rangle = \sqrt{n(n \pm 1)}\,|n \pm 1\rangle. \tag{11}$$

For the Casimir operator, we have

$$(\Gamma_0)^2-(\Gamma_4)^2-T^2=0. \tag{12}$$

The relationship between the algebra and dynamics of this model system is fixed by supposing the Hamiltonian operator is $H=-\frac{1}{2}[N]^{-2}$, so that we have the familiar Coulomb energy spectrum:

$$H|n\rangle=-\frac{1}{2n^2}|n\rangle \qquad n=1,2,\ldots. \tag{13}$$

This algebraic structure may be regarded as a substructure of that appropriate to the three-dimensional case, as discussed in the preceding section. Alternatively, we can take this to be the structure appropriate to the bound states of a 'one-dimensional Coulomb system'. Indeed, we could express the so(2, 1) generators and the Hamiltonian in terms of a single coordinate operator x and a corresponding momentum operator p, by analogy with (1) and (2), but this not essential here.

Normalised SO(2, 1) coherent states are defined by allowing the SO(2, 1) group representatives to act on the ground state $|1\rangle$, and are given by (Perelomov 1972)

$$|z\rangle_P=(1-|z|^2)\sum_{n=1}^{\infty}\sqrt{n}z^{n-1}|n\rangle \qquad z\in\mathbb{C} \qquad |z|<1. \tag{14}$$

In such a state, it is easily checked that

$$\langle N\rangle=\frac{1+|z|^2}{1-|z|^2} \qquad \Delta N=[\langle N^2\rangle-\langle N\rangle^2]^{1/2}=\frac{\sqrt{2}|z|}{1-|z|^2}. \tag{15}$$

The classical limit will correspond here to states with $|z|\to 1$ so that $\langle N\rangle\to\infty$ (think of the correspondence principle). However, we see at once that, as $|z|\to 1$,

$$\frac{\Delta N}{\langle N\rangle}\to\frac{1}{\sqrt{2}} \tag{16}$$

that is, the relative uncertainty in N does not go to zero. This makes the Perelomov coherent states quite unsuitable as initial values of quasiclassical states. The same flaw rules out the Perelomov coherent states in the SO(4, 2) case. This contradicts the statement of Mostowski (1977), that

$$\frac{\Delta N}{\langle N\rangle}\sim\langle N\rangle^{-1/2}$$

for Perelomov coherent states with large $\langle N\rangle$, but we find that statement inconsistent with Mostowski's own definition of those states (his equation (3)), which for suitable choices of parameter values, effectively coincide with the states (14).

Normalised so(2, 1) coherent states are defined as eigenstates of the lowering operator $\Gamma_4-\mathrm{i}T$, and are given by (Barut and Girardello 1971)

$$|z\rangle_{\mathrm{BG}}=\left(\frac{|z|}{I_1(2|z|)}\right)^{1/2}\sum_{n=1}^{\infty}\frac{z^{n-1}}{[n!(n-1)!]^{1/2}}|n\rangle \qquad (\Gamma_4-\mathrm{i}T)|z\rangle_{\mathrm{BG}}=z|z\rangle_{\mathrm{BG}} \tag{17}$$

where z (the eigenvalue of $\Gamma_4-\mathrm{i}T$) can take on any value in the complex plane, and I_1 is the first-order modified Bessel function (Abramowitz and Stegun 1965). In this state, we find

$$\begin{aligned} \langle N\rangle &= \frac{|z|I_0(2|z|)}{I_1(2|z|)} \\ \Delta N &= \left(|z|^2 + \frac{|z|I_0(2|z|)}{I_1(2|z|)} - \frac{|z|^2[I_0(2|z|)]^2}{[I_1(2|z|)]^2}\right)^{1/2} \end{aligned} \tag{18}$$

(where I_0 is the zeroth-order modified Bessel function) and we then deduce from the asymptotic behaviour of the Bessel functions as $|z|\to\infty$,

$$I_0(2|z|) = \frac{\exp(2|z|)}{\sqrt{4\pi|z|}}\left(1 + \frac{1}{16|z|} + \frac{9}{2(16|z|)^2} + \mathrm{O}(|z|^{-3})\right)$$

$$I_1(2|z|) = \frac{\exp(2|z|)}{\sqrt{4\pi|z|}}\left(1 - \frac{3}{16|z|} - \frac{15}{2(16|z|)^2} + \mathrm{O}(|z|^{-3})\right)$$

that $\langle N\rangle \sim |z|$ as $|z|\to\infty$. In this limit, $\Delta N \sim \sqrt{\frac{1}{2}|z|}$, and so

$$\frac{\Delta N}{\langle N\rangle} \sim \frac{1}{\sqrt{2|z|}}$$

so that the relative uncertainty $\Delta N/\langle N\rangle$ goes to zero as $|z|\to\infty$, as desired.

The expectation values and the uncertainties of Γ_4 and T are given by

$$\begin{aligned} &\langle\Gamma_4\rangle = \tfrac{1}{2}(z+z^*) \qquad \langle T\rangle = \tfrac{1}{2}\mathrm{i}(z-z^*) \\ &\Delta\Gamma_4 = \Delta T = (\tfrac{1}{2}\langle\Gamma_0\rangle)^{1/2} = \left(\frac{|z|I_0(2|z|)}{2I_1(2|z|)}\right)^{1/2} \sim (\tfrac{1}{2}|z|)^{1/2}. \end{aligned} \tag{19}$$

We see that when the system is in a Barut–Girardello state, equality holds in the generalised uncertainty relation: $\Delta\Gamma_4\Delta T \geqslant \frac{1}{2}\langle\Gamma_0\rangle$. These states are therefore 'minimum uncertainty states' in this sense. Furthermore, as $|z|\to\infty$, (19) shows that

$$\frac{(\Delta\Gamma_4)^2 + (\Delta T)^2}{\langle\Gamma_4\rangle^2 + \langle T\rangle^2} \to 0$$

as desired.

The Barut–Girardello states evolve under Schrödinger time evolution as

$$|\psi(t)\rangle = \exp(-\mathrm{i}Ht)|\psi(0)\rangle = \left(\frac{|z|}{I_1(2|z|)}\right)^{1/2} \sum_{n=1}^{\infty} \frac{z^{n-1}}{[n!(n-1)!]^{1/2}} \exp\left(\frac{\mathrm{i}t}{2n^2}\right)|n\rangle \tag{20}$$

where we have set $|\psi(0)\rangle = |z\rangle_{\mathrm{BG}}$. Note that $|\psi(t)\rangle$ is not a Barut–Girardello state for $t>0$ unless $z=0$. The expectation value and the uncertainty of the constant of the motion Γ_0 are of course constant in the state $|\psi(t)\rangle$. The expectation values of the operators Γ_4 and T at time t are found from (20) to be

$$\begin{aligned} \langle\Gamma_4\rangle &= \frac{|z|}{I_1(2|z|)} \sum_{n=1}^{\infty} \frac{|z|^{2n-2}}{n!(n-1)!}(xC_n(t) + yS_n(t)) \\ \langle T\rangle &= \frac{|z|}{I_1(2|z|)} \sum_{n=1}^{\infty} \frac{|z|^{2n-2}}{n!(n-1)!}(xS_n(t) - yC_n(t)) \\ C_n(t) &= \cos\left(\frac{t(n+\frac{1}{2})}{n^2(n+1)^2}\right) \qquad S_n(t) = \sin\left(\frac{t(n+\frac{1}{2})}{n^2(n+1)^2}\right) \qquad z = x+\mathrm{i}y \qquad x, y \in \mathbb{R}. \end{aligned} \tag{21}$$

Furthermore, we find

$$\langle(\Gamma_4 - \mathrm{i}T)^2\rangle = \frac{|z|}{I_1(2|z|)} \sum_{n=1}^{\infty} \frac{z^2 \cdot |z|^{2n+2}}{n!(n-1)!} \exp\left(\frac{-2\mathrm{i}t(n+1)}{n^2(n+2)^2}\right) \tag{22}$$

together with its complex conjugate, so that

$$\langle(\Gamma_4)^2\rangle = \frac{1}{2}\left(|z|^2 + \frac{|z| I_0(2|z|)}{I_1(2|z|)} + \frac{|z|}{I_1(2|z|)} \sum_{n=1}^{\infty} \frac{|z|^{2n-2}}{n!(n-1)!}[(x^2-y^2)c_n(t) + 2xys_n(t)]\right)$$

$$\langle T^2\rangle = \frac{1}{2}\left(|z|^2 + \frac{|z| I_0(2|z|)}{I_1(2|z|)} - \frac{|z|}{I_1(2|z|)} \sum_{n=1}^{\infty} \frac{|z|^{2n-2}}{n!(n-1)!}[(x^2-y^2)c_n(t) + 2xys_n(t)]\right) \tag{23}$$

$$c_n(t) = \cos\left(\frac{2t(n+1)}{n^2(n+2)^2}\right) \qquad s_n(t) = \sin\left(\frac{2t(n+1)}{n^2(n+2)^2}\right)$$

where we have used (12). Thus

$$\langle(\Gamma_4)^2\rangle + \langle T^2\rangle = |z|^2 + \frac{|z| I_0(2|z|)}{I_1(2|z|)} \tag{24}$$

at all times, and therefore

$$(\Delta\Gamma_4)^2 + (\Delta T)^2 = |z|^2 + \frac{|z| I_0(2|z|)}{I_1(2|z|)} - \langle\Gamma_4\rangle^2 - \langle T\rangle^2. \tag{25}$$

The expressions for the individual expectation values and the uncertainties of Γ_4 and T are intractable, but we are only interested in the asymptotic behaviour as $|z| \to \infty$. We find from (21) and (22),

$$\begin{aligned}\langle\Gamma_4 - \mathrm{i}T\rangle &= \frac{z}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}v^2 - \frac{\mathrm{i}t}{|z|^3}\left(1 - \frac{3v}{\sqrt{2|z|}} + \frac{3v^2 - \frac{3}{2}}{|z|} + \mathrm{O}(|z|^{-3/2})\right)\right] \mathrm{d}v \\ \langle(\Gamma_4 - \mathrm{i}T)^2\rangle &= \frac{z^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}v^2 - \frac{2\mathrm{i}t}{|z|^3}\left(1 - \frac{3v}{\sqrt{2|z|}} + \frac{3v^2 - 3}{|z|} + \mathrm{O}(|z|^{-3/2})\right)\right] \mathrm{d}v.\end{aligned} \tag{26}$$

To obtain these results, we have introduced, for each value of z, and corresponding Barut–Girardello state, the operator

$$V(z) = \frac{N - |z|}{\sqrt{\frac{1}{2}|z|}}$$

with expectation value asymptotically approaching 0 and uncertainty approaching 1 as $|z| \to \infty$. Eigenvalues of $V(z)$ are labelled v in (26). In the asymptotic limit, the probability distribution on the eigenvalues of $V(z)$ can be shown to be standard normal. The time dependence in the exponential in (26) then follows from (21) and (22).

For times t of order $|z|^3$, i.e., times of the order of the classical period $\tau = 2\pi(-2\langle H\rangle)^{-3/2} = 2\pi|z|^3$, we put $t = \xi|z|^3$ with $\xi = \mathrm{O}(1)$, so that we have from (26)

$$\langle\Gamma_4 - \mathrm{i}T\rangle = z\exp(-\mathrm{i}\xi) + \mathrm{O}(\sqrt{|z|}) \qquad \langle(\Gamma_4 - \mathrm{i}T)^2\rangle = z^2 \exp(-2\mathrm{i}\xi) + \mathrm{O}(|z|^{3/2}). \tag{27}$$

It can then be shown from the constancy of $\langle(\Gamma_4)^2\rangle + \langle T^2\rangle = \langle(\Gamma_0)^2\rangle$ that, for times of this order, $\Delta\Gamma_4 = \sqrt{\frac{1}{2}|z|} + \mathrm{O}(1)$, $\Delta T = \sqrt{\frac{1}{2}|z|} + \mathrm{O}(1)$. The relative uncertainties of Γ_4 and T therefore remain small and constant for times of this order. Also, according to (27), the expectation values of Γ_4 and T follow the corresponding values for a classical trajectory: in fact, for times $t = \mathrm{o}(|z|^{7/2})$, we have

$$\langle\Gamma_4(t) \pm \mathrm{i}T(t)\rangle \sim \langle\Gamma_4(0) \pm \mathrm{i}T(0)\rangle \exp\left(\frac{\pm\mathrm{i}t}{|z|^3}\right).$$

The point in the plane with Cartesian coordinates ($\langle\Gamma_4\rangle$, $\langle T\rangle$) moves around a circle, with the classical period τ (compare with the three-dimensional case, figure 1). The states we have constructed for the one-dimensional Coulomb system can therefore be regarded as quasicalssical for times of the order of the classical period ($t = O(|z|^3)$).

For time t of order $|z|^{7/2}$, we put $t = \sigma|z|^{7/2}$ with $\sigma = O(1)$ and see that

$$\begin{aligned}\langle\Gamma_4 - \mathrm{i}T\rangle &= z\exp(-\tfrac{9}{4}\sigma^2 - \mathrm{i}\sigma\sqrt{|z|})(1 + O(|z|^{-1/2})) \\ \langle(\Gamma_4 - \mathrm{i}T)^2\rangle &= z^2\exp(-9\sigma^2 - 2\mathrm{i}\sigma\sqrt{|z|})(1 + O(|z|^{-1/2}))\end{aligned} \tag{28}$$

so that

$$\begin{aligned}&\langle\Gamma_4\rangle \sim \exp(-\tfrac{9}{4}\sigma^2)(x\cos(\sigma\sqrt{|z|}) + y\sin(\sigma\sqrt{|z|})) \\ &\langle T\rangle \sim \exp(-\tfrac{9}{4}\sigma^2)(x\sin(\sigma\sqrt{|z|}) - y\cos(\sigma\sqrt{|z|})) \\ &\langle(\Gamma_4)^2\rangle \sim \frac{1}{2}\left(|z|^2 + \frac{|z|I_0(2|z|)}{I_1(2|z|)} + \exp(-9\sigma^2)[(x^2 - y^2)\cos(2\sigma\sqrt{|z|}) + 2xy\sin(2\sigma\sqrt{|z|})]\right) \\ &\langle T^2\rangle \sim \frac{1}{2}\left(|z|^2 + \frac{|z|I_0(2|z|)}{I_1(2|z|)} - \exp(-9\sigma^2)[(x^2 - y^2)\cos(2\sigma\sqrt{|z|}) + 2xy\sin(2\sigma\sqrt{|z|})]\right).\end{aligned} \tag{29}$$

It can now be seen that for times of this order, the point with Cartesian coordinates ($\langle\Gamma_4\rangle, \langle T\rangle$) continues to revolve in the plane around the origin at a constant angular velocity. However, its distance from the origin decays as a Gaussian function of time. On the same timescale, the uncertainties of Γ_4 and T increase until $(\Delta\Gamma_4)^2 + (\Delta T)^2$ attains its maximum possible value of $|z|^2 + |z|I_0(2|z|)/I_1(2|z|)$. Thus the state spreads over times of order $|z|^{7/2}$, i.e. of order $\tau^{7/6}$.

It may appear that we now have the whole story regarding the asymptotic time-dependence of $\langle\Gamma_4\rangle$ and $\langle T\rangle$ but this not the case. There is also some surprising behaviour for times of order $|z|^4$. If J is an integer and the time differs from $\frac{2}{3}\pi J|z|^4$ by an interval of order $|z|^{7/2}$, then $t = \frac{2}{3}\pi J|z|^4 + \sigma|z|^{7/2}$ with $\sigma = O(1)$, and the phase in $C_n(t)$ and $S_n(t)$ of (21) for $n = |z| + O(\sqrt{|z|})$ varies by approximately $2\pi J$ between successive values of n. Successive contributions therefore tend to reinforce each other. After accounting for the appropriate integral multiples of 2π in the phases, we get

$$\langle\Gamma_4 - \mathrm{i}T\rangle \sim \frac{(-1)^J z}{\sqrt{1 + \mathrm{i}4\pi J}}\exp\left(-\mathrm{i}\frac{8\pi J|z|}{3} - \mathrm{i}\sigma\sqrt{|z|} - \frac{9\sigma^2(1 - \mathrm{i}4\pi J)}{4(1 + 16\pi^2J^2)}\right)(1 + O(|z|^{-1/2})) \tag{30}$$

so that ($\langle\Gamma_4\rangle, \langle T\rangle$) has phase

$$\Phi = \pi J - \arg z + \frac{8\pi J|z|}{3} + \sigma\sqrt{|z|} + \tfrac{1}{2}\tan^{-1}(4\pi J) - \frac{9\pi J\sigma^2}{1 + 16\pi^2J^2} + O(|z|^{-1/2}) \tag{31}$$

and magnitude

$$R = \frac{|z|}{(1 + 16\pi^2J^2)^{1/4}}\exp\left(\frac{-9\sigma^2}{4(1 + 16\pi^2J^2)}\right)(1 + O(|z|^{-1/2})). \tag{32}$$

Therefore $\langle\Gamma_4\rangle$ and $\langle T\rangle$ become significant at times near $\frac{2}{3}\pi J|z|^4 (= \frac{1}{3}J(\tau^4/2\pi)^{1/3})$ when the original coherence tries to 'reassert' itself. The peak magnitude is given by

$$R_{\max} = \frac{|z|}{(1 + 16\pi^2J^2)^{1/4}} \sim \frac{|z|}{2\sqrt{\pi J}} \tag{33}$$

so that successive 'resurgences of coherence' decrease in strength. Furthermore, the characteristic time occupied by the Jth resurgence increases with J, being proportional to $\sqrt{1+16\pi^2 J^2}(\sim 4\pi J)$.

These results can be summarised as follows. The states are quasiclassical for times of the order of the classical period τ, and the expectation values of the non-constant operators follow the classically predicted trajectory to within a factor of $O(|z|^{-1/2})$. The uncertainties remain constant to within a similar factor. For times of order $\tau^{7/6}$, the expectation values of the non-constant operators decay to zero and their uncertainties increase until the sum of their squares reaches the maximum possible value. The uncertainties become of the order of $\langle\Gamma_0\rangle \sim |z|$, so that for times of this order, the states are no longer quasiclassical. Effectively, the states 'spread' around the classical orbit. A surprising feature is that the states partially 'reassert their coherence', with the expectation values of non-constant operators diverging from zero, at regular time intervals of length $\frac{1}{3}(\tau^4/2\pi)^{1/3}$. We can also show that each of the uncertainties $\Delta\Gamma_4$ and ΔT diverges from its limiting value of $[\frac{1}{2}(|z|^2+|z|I_0(2|z|)/I_1(2|z|))]^{1/2}$ at regular time intervals of $\frac{1}{3}\pi|z|^4(=\frac{1}{6}(\tau^4/2\pi)^{1/3})$.

These results are illustrated for the case $z=10\,000$ in figures 2-6 obtained by numerical evaluation of (21) and (22). For ease of comparison, the range on the vertical axis is from $-11\,000$ to $11\,000$ in each case. Figure 2 shows the behaviour of $\langle\Gamma_4\rangle$ and $\langle\Gamma_4\rangle \pm \Delta\Gamma_4$ for three classical periods from $t=0$, and figure 3 shows the behaviour of $\langle T\rangle$ and $\langle T\rangle \pm \Delta T$ for the same interval. The classical values have not been marked since they are virtually indistinguishable from the expectation values over this interval. Note that the uncertainties remain small but that they do actually grow substantially over the interval. The reason for this is that the behaviour at times of order $|z|^{7/2}$ has a significant effect even for these small times. Note also the drops in the uncertainties of Γ_4 and T when the expectation values reach their maxima and minima. These drops can be seen to arise from the fact that the state spreads around

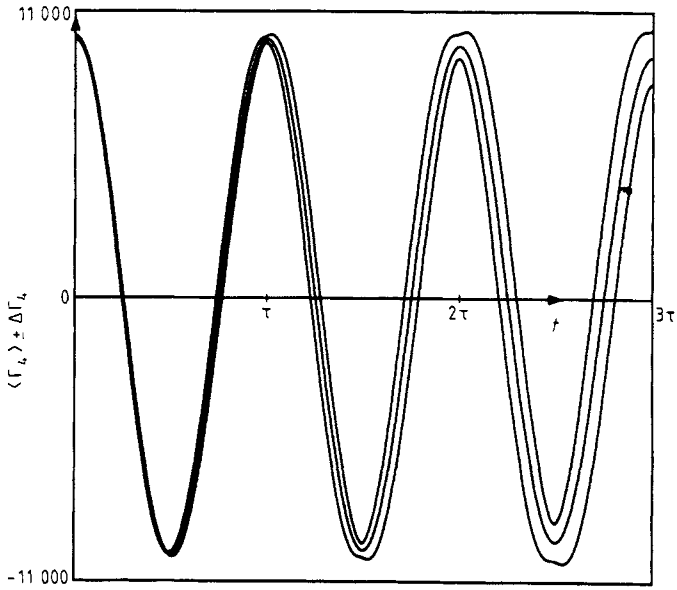

Figure 2. Behaviour of $\langle\Gamma_4\rangle$ and $\langle\Gamma_4\rangle \pm \Delta\Gamma_4$ with $z=10\,000$ for three classical periods τ from time $t=0$.

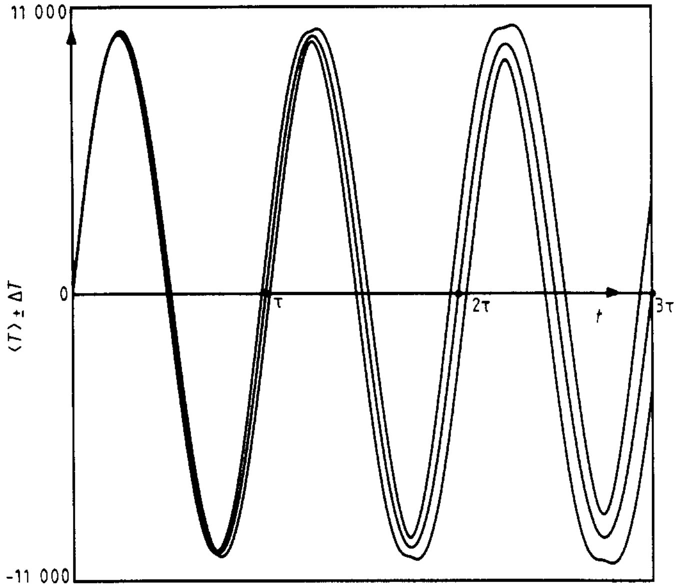

Figure 3. Behaviour of $\langle T\rangle$ and $\langle T\rangle \pm \Delta T$ with $z = 10\,000$ for three classical periods τ from time $t = 0$.

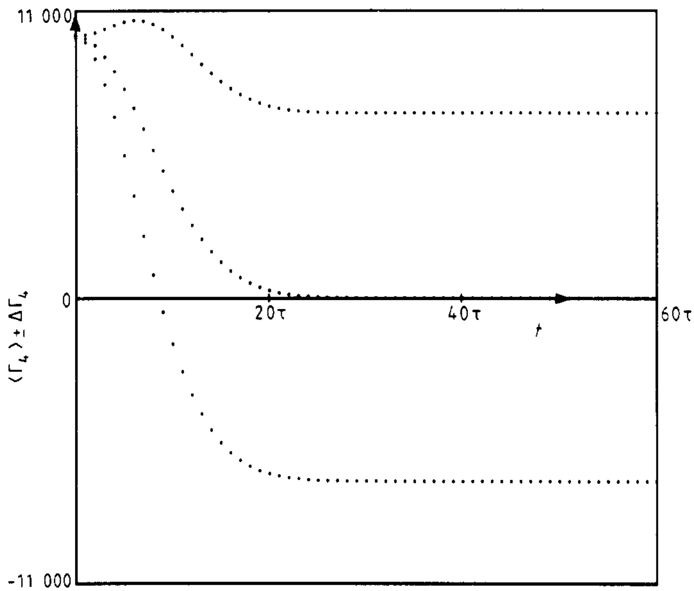

Figure 4. Behaviour of $\langle \Gamma_4\rangle$ and $\langle \Gamma_4\rangle \pm \Delta\Gamma_4$ with $z = 10\,000$ for 60 classical periods τ from time $t = 0$.

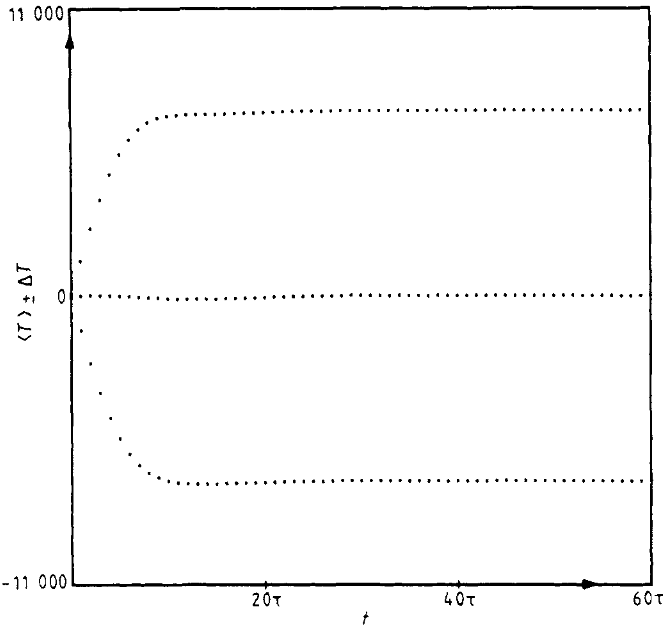

Figure 5. Behaviour of $\langle T\rangle$ and $\langle T\rangle \pm \Delta T$ with $z = 10\,000$ for 60 classical periods τ from time $t = 0$.

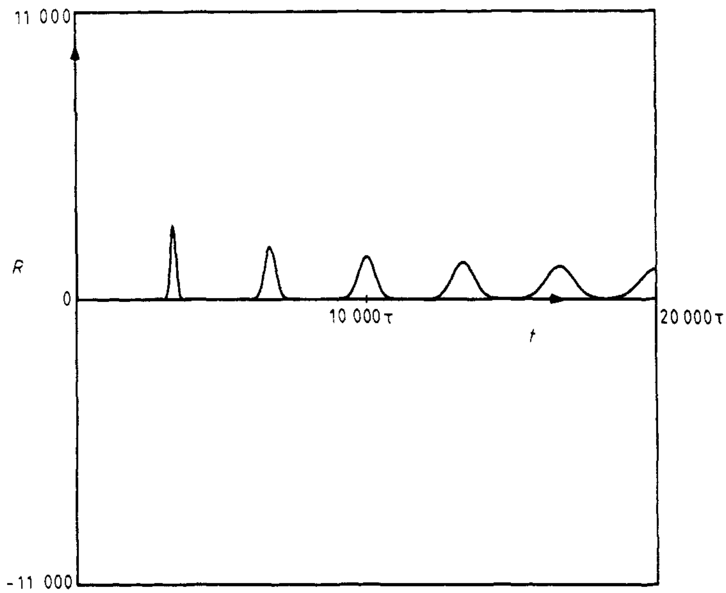

Figure 6. Behaviour of $R = \sqrt{\langle\Gamma_4\rangle^2 + \langle T\rangle^2}$ with $z = 10\,000$ for 20 000 classical periods τ from time $t = 0$.

the orbit, so that there is an angular spread in $(\langle\Gamma_4\rangle, \langle T\rangle)$-space but no spread in the distance of $(\langle\Gamma_4\rangle, \langle T\rangle)$ from the origin.

Figure 4 shows the behaviour of $\langle\Gamma_4\rangle$ and $\langle\Gamma_4\rangle \pm \Delta\Gamma_4$ for 60 classical periods from $t=0$ evaluated at regular intervals of a single period and figure 5 shows the behaviour of $\langle T\rangle$ and $\langle T\rangle \pm \Delta T$ for the same interval, also at regular intervals of a single period. Note that $\langle\Gamma_4\rangle$ appears to decay away (along a bell-shaped curve) after about 20 classical periods when, according to (29), $(\langle\Gamma_4\rangle^2 + \langle T\rangle^2)^{1/2}$ should be about 3% of its initial value. Note also that the uncertainties appear to level off after about 12 classical periods at which time their increase should be about 96% of the total increase. The other apparent pattern, that $\langle T\rangle$ appears to stay near zero, is a consequence solely of the fact that the evaluations have been made at integral multiples of the classical period (to bring out the Gaussian functional dependence in $\langle\Gamma_4\rangle$). In fact, $\langle T\rangle$ varies significantly away from zero between the evaluation times.

Figure 6 shows the behaviour of $R = (\langle\Gamma_4\rangle^2 + \langle T\rangle^2)^{1/2}$ for 20 000 classical periods from $t=0$. The behaviour described in (32) can now be seen to occur over this interval, and at the specific times discussed in the context of that equation. As can be seen from the figure, the sixth divergence of $(\langle\Gamma_4\rangle^2 + \langle T\rangle^2)^{1/2}$ from zero begins just after the fifth divergence has finished (and each successive divergence will begin before the previous one has finished). This arises because the process of divergence shows down each time and is associated with successively longer characteristic times: by the time of the fifth and sixth divergences, the characteristic times are of the same order as the regular interval (about 3300 classical periods).

One further feature should be noted for the states we have constructed. Despite the spreading of these states, over times of order $|z|^{7/2}$ when $|z|$ is large, all of the states are in fact quasiperiodic: let α be an integer so large that

$$\frac{|z|}{I_1(2|z|)} \sum_{n=\alpha+1}^{\infty} \frac{|z|^{2n-2}}{n!(n-1)!} < \epsilon$$

where ϵ is small and positive, and let $t = 4\pi\eta^2$ where η is an integer divisible by all integers less than or equal to α. Then

$$\langle n|\psi(t)\rangle = \langle n|\psi(0)\rangle$$

for $n \leq \alpha$, and so $\||\psi(t)\rangle - |\psi(0)\rangle\| < 2\sqrt{\epsilon}$. The expectation values $\langle\Gamma_4\rangle$ and $\langle T\rangle$ therefore approach their initial values to within $2\sqrt{\epsilon}|z|$. The times involved can be seen to be extremely large, since $\ln \eta = \alpha + o(\alpha)$ (Apostol 1976). In particular, for $|z|$ large and at times that are integral multiples of $4\pi\eta^2$ where η is an integer divisible by all integers less than or equal to $|z| + \kappa\sqrt{|z|} + 1$, then $\langle n|\psi(t)\rangle = \langle n|\psi(0)\rangle$ provided $n \leq |z| + \kappa\sqrt{|z|} + 1$. Therefore

$$\Re\{\langle\psi(t)|\psi(0)\rangle\} \geq \operatorname{erf}\kappa + O(|z|^{-1/2})$$

(where erf is the error function $\operatorname{erf} x = 2/\sqrt{\pi}\int_0^x \exp(-t^2)\,dt$), by which is meant that there exist $\mu_0 > 0$ and $\delta > 0$ such that

$$\Re\{\langle\psi(t)|\psi(0)\rangle\} \geq \operatorname{erf}\kappa - \delta|z|^{-1/2} \qquad \text{if } |z| > \mu_0.$$

Furthermore,

$$\Re\left(\frac{\langle\Gamma_4 - \mathrm{i}T\rangle}{z}\right) \geq \operatorname{erf}\kappa + O(|z|^{-1/2})$$

so that the expectation values of Γ_4 and T approach their initial values arbitrarily closely and their uncertainties approach their initial (and minimum possible) values arbitrarily closely.

In fact, we do not require quite such extreme times for quasiperiodicity. If we take η to be an integer divisible by all integers greater than or equal to $|z| - \kappa\sqrt{|z|} - 1$ and less than or equal to $|z| + \kappa\sqrt{|z|} + 1$, then

$$\Re\{\langle\psi(t)|\psi(0)\rangle\} \geq 2\,\mathrm{erf}\,\kappa - 1 + \mathrm{O}(|z|^{-1/2})$$

and

$$\Re\left(\frac{\langle\Gamma_4 - \mathrm{i}T\rangle}{z}\right) \geq 2\,\mathrm{erf}\,\kappa - 1 + \mathrm{O}(|z|^{-1/2}).$$

In this case,

$$\ln \eta = 2\kappa\sqrt{|z|}\,\ln|z|(1 + \mathrm{o}(1))$$

but the time required for quasiperiodicity is typically still extremely large. To demonstrate the extremity of the times involved, it is instructive to illustrate all the time intervals that have been discussed. For the hydrogen atom, the unit of time ($\hbar^3/me^4$) is equal to about 2.4×10^{-17} seconds. The classical period corresponding to $z = 10\,000$ is then about 1.5×10^{-4} seconds. The characteristic time for 'loss of coherence' for $z = 10\,000$ is about 1.6×10^{-3} seconds, and the regular interval for the 'departure from incoherence' is about half a second. The time required for quasiperiodicity, that is, for expectation values to approach their original values to within $10^{-3}|z|$, is in the region of 10^{4000} times the age of the universe.

4. Quasiclassical states for the three-dimensional Coulomb system

The three-dimensional case is completely analogous to the one-dimensional case but the corresponding calculations are naturally more complicated. As in the case of so(2, 1), the uncertainty of the principal quantum number of Perelomov states is of the same order as its expectation value in the asymptotic limit as $\langle N\rangle \to \infty$, so we will work with the Barut–Girardello states. We start with the orthonormal basis

$$\{|n, m, p\rangle\colon n = 1, 2, \ldots;\ |m| + |p| + 1 \leq n;\ n + m + p \text{ odd}\}.$$

These vectors are eigenvectors of N, L_3, A_3 corresponding to eignevalues n, m, p respectively. The phases have been chosen so that the matrix elements of the remaining so(4, 2) operators $L_\pm + A_\pm$, $L_\pm - A_\pm$, etc. (where $L_\pm = L_1 \bullet \mathrm{i}L_2$, $A_\pm = A_1 \pm \mathrm{i}A_2$, etc.), are given by

$$(L_- + A_-)|n, m, p\rangle = [(n - m - p + 1)(n + m + p - 1)]^{1/2}|n, m - 1, p - 1\rangle$$

$$(L_+ + A_+)|n, m, p\rangle = [(n - m - p - 1)(n + m + p + 1)]^{1/2}|n, m + 1, p + 1\rangle$$

$$(L_- - A_-)|n, m, p\rangle = [(n - m + p + 1)(n + m - p - 1)]^{1/2}|n, m - 1, p + 1\rangle$$

$$(L_+ - A_+)|n, m, p\rangle = [(n - m + p - 1)(n + m - p + 1)]^{1/2}|n, m + 1, p - 1\rangle$$

$$(M_+ + \mathrm{i}\Gamma_+)|n, m, p\rangle = [(n - m - p - 1)(n - m + p - 1)]^{1/2}|n - 1, m + 1, p\rangle$$

$$
\begin{aligned}
(M_- - \mathrm{i}\Gamma_-)|n, m, p\rangle &= [(n-m-p+1)(n-m+p+1)]^{1/2}|n+1, m-1, p\rangle \\
(M_- + \mathrm{i}\Gamma_-)|n, m, p\rangle &= -[(n+m-p-1)(n+m+p-1)]^{1/2}|n-1, m-1, p\rangle
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
(M_+ - \mathrm{i}\Gamma_+)|n, m, p\rangle &= -[(n+m-p+1)(n+m+p+1)]^{1/2}|n+1, m+1, p\rangle \\
(\Gamma_4 - \mathrm{i}T - \mathrm{i}\Gamma_3 - M_3)|n, m, p\rangle &= -[(n+m-p-1)(n-m-p-1)]^{1/2}|n-1, m, p+1\rangle \\
(\Gamma_4 + \mathrm{i}T + \mathrm{i}\Gamma_3 - M_3)|n, m, p\rangle &= -[(n+m-p+1)(n-m-p+1)]^{1/2}|n+1, m, p-1\rangle \\
(\Gamma_4 - \mathrm{i}T + \mathrm{i}\Gamma_3 + M_3)|n, m, p\rangle &= [(n+m+p-1)(n-m+p-1)]^{1/2}|n-1, m, p-1\rangle \\
(\Gamma_4 + \mathrm{i}T - \mathrm{i}\Gamma_3 + M_3)|n, m, p\rangle &= [(n+m+p+1)(n-m+p+1)]^{1/2}|n+1, m, p+1\rangle.
\end{aligned}
$$

The general Barut-Girardello state vector is found by diagonalising the commuting lowering operators $\Gamma_4 - \mathrm{i}T$, $\mathbf{\Gamma} - \mathrm{i}\boldsymbol{M}$, or equivalently

$$
\begin{aligned}
a_1 &= \tfrac{1}{2}(\mathrm{i}\Gamma_1 + M_1 - \Gamma_2 + \mathrm{i}M_2) \\
a_2 &= \tfrac{1}{2}(\mathrm{i}\Gamma_3 + M_3 - \Gamma_4 + \mathrm{i}T) \\
b^1 &= \tfrac{1}{2}(-\mathrm{i}\Gamma_1 - M_1 - \Gamma_2 + \mathrm{i}M_2) \\
b^2 &= \tfrac{1}{2}(-\mathrm{i}\Gamma_3 - M_3 - \Gamma_4 + \mathrm{i}T).
\end{aligned} \tag{35}
$$

From the representation relations (9), it can be shown that $a_1 b^1 + a_2 b^2 = 0$, so that the eigenvalues z_1, z_2, ζ_1, ζ_2 of a_1, a_2, b^1, b^2, respectively, satisfy $z_1\zeta_1 + z_2\zeta_2 = 0$. The Barut-Girardello states are found to be

$$
|z_1, z_2; \zeta_1, \zeta_2\rangle_{\mathrm{BG}} = \sum_{n=1}^{\infty} \sum_{m=-(n-1)}^{n-1} \sideset{}{'}\sum_{p=-(n-1-|m|)}^{n-1-|m|} \frac{P(n, m, p, z_1, z_2, \zeta_1, \zeta_2)}{\sqrt{A(n, m, p)}} |n, m, p\rangle \tag{36}
$$

where Σ' indicates that p must be summed over over integers of the same parity as $n-1-|m|$, and

$$
\begin{aligned}
A(n, m, p) &= q!(q-p)!(q-m)!(q-m-p)! \\
q &= \tfrac{1}{2}(n+m+p-1), \\
P(n, m, p, z_1, z_2, \zeta_1, \zeta_2) &= \zeta_1^m z_2^{(n-m-p-1)/2}(-\zeta_2)^{(n-m+p-1)/2} \\
&= z_1^{-m} z_2^{(n+m-p-1)/2}(-\zeta_2)^{(n+m+p-1)/2} \\
&= z_1^{(n-m-p-1)/2} \zeta_1^{(n+m-p-1)/2}(-\zeta_2)^p \\
&= z_1^{(n-m+p-1)/2} \zeta_1^{(n+m+p-1)/2} z_2^{-p}.
\end{aligned} \tag{37}
$$

More precisely, $P(n, m, p, z_1, z_2, \zeta_1, \zeta_2)$ is chosen equal to whichever of these four expressions is sensible (at least one is sensible if at least one of the eigenvalues is non-zero). In the case that $|z_1| = |z_2| = |\zeta_1| = |\zeta_2| = 0$, then $|0, 0; 0, 0\rangle = |1, 0, 0\rangle$, the ground state. The norm of the state vector is given by the square root of

$$
{}_{\mathrm{BG}}\langle z_1, z_2; \zeta_1, \zeta_2 | z_1, z_2; \zeta_1, \zeta_2\rangle_{\mathrm{BG}} = I_0(2\mu) \tag{38}
$$

where $\mu = [|z_1|^2 + |z_2|^2 + |\zeta_1|^2 + |\zeta_2|^2]^{1/2}$. For example, when neither z_1 nor ζ_1 is zero,

$$
{}_{\mathrm{BG}}\langle z_1, z_2; \zeta_1, \zeta_2 | z_1, z_2; \zeta_1, \zeta_2\rangle_{\mathrm{BG}} = \sum_{q=-\infty}^{\infty} \left(\frac{|z_1|^2}{|z_2\zeta_2|}\right)^q I_q(2|z_2|) I_q(2|\zeta_2|) = I_0(2\mu) \tag{39}
$$

the final equality following from Graf's addition theorem for Bessel functions (Abramowitz and Stegun 1965, Watson 1922).

When the system is prepared in the state $|z_1, z_2, \zeta_1, \zeta_2\rangle_{\mathrm{BG}}$ then the expectation value and the uncertainty of the constant operator N are given by

$$
\begin{aligned}
&\langle N\rangle = \lambda(\mu)+1 \qquad \Delta N = \sqrt{\mu^2-\lambda(\mu)^2}\\
&\lambda(\mu) = \frac{\mu I_1(2\mu)}{I_0(2\mu)}.
\end{aligned}
\tag{40}
$$

From the asymptotic expansions of I_0 and I_1, then as $\mu\to\infty$,

$$
\langle N\rangle = \mu+\tfrac{3}{4}+\mathrm{O}(\mu^{-1}) \qquad \Delta N = \sqrt{\frac{\mu}{2}}+\mathrm{O}(\mu^{-1/2}).
\tag{41}
$$

The asymptotic expansion of the relative uncertainty is now given by

$$
\frac{\Delta N}{\langle N\rangle} = \frac{1}{\sqrt{2\mu}}+\mathrm{O}(\mu^{-3/2})
\tag{42}
$$

which decays to 0 as $\mu\to\infty$. It can also be shown that

$$
\begin{aligned}
&\langle L_3\rangle = (|\zeta_1|^2-|z_1|^2)\frac{\lambda(\mu)}{\mu^2}\\
&\langle (L_3)^2\rangle = \frac{-I_2(2\mu)}{\mu^2 I_0(2\mu)}\{(|z_1|^2+|\zeta_1|^2)(|z_2|^2+|\zeta_2|^2)+4|z_1z_2\zeta_1\zeta_2|\}+|z_1|^2+|\zeta_1|^2
\end{aligned}
\tag{43}
$$

where $\lambda(\mu)$ is as before, with similar results for $\langle A_3\rangle$ and $\langle (A_3)^2\rangle$. The asymptotic behaviour of $I_2(2\mu)$ is given by (Abramowitz and Stegun 1965)

$$
I_2(2\mu) = \frac{\exp(2\mu)}{\sqrt{4\pi\mu}}\left(1-\frac{15}{16\mu}+\mathrm{O}(\mu^{-2})\right)
\tag{44}
$$

so that as $\mu\to\infty$, for example,

$$
\begin{aligned}
&\langle L_3\rangle = (|\zeta_1|^2-|z_1|^2)(\mu^{-1}-\tfrac{1}{4}\mu^{-2}+\mathrm{O}(\mu^{-3}))\\
&\langle (L_3)^2\rangle = \mu^{-3}\{(|z_1|^2+|\zeta_1|^2)(\tfrac{1}{2}|z_1|^2+\tfrac{1}{2}|\zeta_1|^2+|z_2|^2+|\zeta_2|^2)+2|z_1z_2\zeta_1\zeta_2|\}+\mathrm{O}(1).
\end{aligned}
\tag{45}
$$

Since $|z_1|$, $|z_2|$, $|\zeta_1|$, $|\zeta_2|$ are all of order μ as $\mu\to\infty$, then $\langle L_4\rangle$, $\langle A_3\rangle$ are of order μ and ΔL_3, ΔA_3 are of order $\mu^{1/2}$. Similarly, $\langle L_1\rangle$, $\langle L_2\rangle$, $\langle A_1\rangle$, $\langle A_2\rangle$, are also of order μ as $\mu\to\infty$, for example,

$$
\langle L_1+\mathrm{i}L_2+A_1+\mathrm{i}A_2\rangle = \frac{2\lambda(\mu)}{\mu^2}(\zeta_1^*z_2-z_1\zeta_2^*) = 2(\mu^{-1}-\tfrac{1}{4}\mu^{-2}+\mathrm{O}(\mu^{-3}))(\zeta_1^*z_2-z_1\zeta_2^*).
\tag{46}
$$

Furthermore, it is not hard to see that ΔL_1, ΔL_2, ΔA_1, ΔA_2, are of order $\mu^{1/2}$. Because the uncertainties of $\boldsymbol{L}$, $\boldsymbol{A}$, N are of order $\mu^{1/2}$ and their expectation values are of order μ as $\mu\to\infty$, the state is concentrated onto the corresponding classical orbit, at least in so(4, 2) variable space, as that limit is approached. By this, we mean that if we take the formal limit as $\mu\to\infty$, $\hbar\to 0$ such that $\hbar\mu$ is kept constant, then the particle is found on the corresponding orbit with certainty. This implies that the Barut–Girardello states and their time evolution give a quasiclassical approximation to the orbit, at least as far as the *constant* operators are concerned.

We now consider the expectation values and uncertainties of the non-constant operators $(\boldsymbol{\Gamma}, \boldsymbol{M}, \Gamma_4, T)$. These can be calculated in closed form at $t=0$. The expectation values at $t=0$ can be calculated from (35) and

$$\langle a_1\rangle = z_1 \qquad \langle a_2\rangle = z_2 \qquad \langle b^1\rangle = \zeta_1 \qquad \langle b^2\rangle = \zeta_2 \tag{47}$$

so that $\langle\boldsymbol{\Gamma}\rangle^2+\langle\boldsymbol{M}\rangle^2+\langle\Gamma_4\rangle^2+\langle T\rangle^2=2\mu^2$. The uncertainties at $t=0$ are now given by

$$\Delta M_1=\Delta M_2=\Delta M_3=\Delta\Gamma_1=\Delta\Gamma_2=\Delta\Gamma_3=\Delta T=\Delta\Gamma_4=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\lambda(\mu)}. \tag{48}$$

Thus the uncertainties are of order $\mu^{1/2}$, and the state is quasiclassical at $t=0$ (localised in so(4, 2) variable space as $\mu\to\infty$).

From the constancy of the operators $(M_1)^2+(\Gamma_1)^2$, $(M_2)^2+(\Gamma_2)^2$, $(M_3)^2+(\Gamma_3)^2$, $T^2+(\Gamma_4)^2$, which follow from (9), we have at all times,

$$\langle(M_1)^2\rangle+\langle(\Gamma_1)^2\rangle=|z_1-\zeta_1|^2+\lambda(\mu)+1 \tag{49}$$

with similar expressions for $\langle(M_2)^2\rangle+\langle(\Gamma_2)^2\rangle$, $\langle(M_3)^2\rangle+\langle(\Gamma_3)^2\rangle$, $\langle T^2\rangle+\langle(\Gamma_4)^2\rangle$. Therefore the four expressions $(\Delta M_1)^2+(\Delta\Gamma_1)^2$, $(\Delta M_2)^2+(\Delta\Gamma_2)^2$, $(\Delta M_3)^2+(\Delta\Gamma_3)^2$, $(\Delta T)^2+(\Delta\Gamma_4)^2$ are bounded above. Similarly, by the generalised uncertainty relations, the four products $(\Delta M_1)^2(\Delta\Gamma_1)^2$, $(\Delta M_2)^2(\Delta\Gamma_2)^2$, $(\Delta M_3)^2(\Delta\Gamma_3)^2$, $(\Delta T)^2(\Delta\Gamma_4)^2$ are bounded below. For example,

$$(\Delta M_1)^2(\Delta\Gamma_1)^2\geqslant\tfrac{1}{4}\langle N\rangle^2=\tfrac{1}{4}(\lambda(\mu)+1)^2 \tag{50}$$

and similarly for the other products, so that the sums are all also bounded below:

$$(\Delta M_1)^2+(\Delta\Gamma_1)^2\geqslant\lambda(\mu)+1 \tag{51}$$

and similarly for the other sums. These lower bounds are attained at $t=0$.

The time evolution of the Barut–Girardello state is given by

$$\begin{aligned}|\psi(t)\rangle &= \sum_{n=1}^{\infty}\sum_{m=-(n-1)}^{n-1}\sideset{}{'}\sum_{p=-(n-1-|m|)}^{n-1-|m|} K(n,m,p,t)|n,m,p\rangle\\ K(n,m,p,t) &= \frac{P(n,m,p,z_1,z_2,\zeta_1,\zeta_2)}{\sqrt{A(n,m,p)}}\exp\left(\frac{\mathrm{i}t}{2n^2}\right)\end{aligned} \tag{52}$$

where the notation is as in (37), and we have put $|\psi(0)\rangle=|z_1,z_2;\zeta_1,\zeta_2\rangle_{\mathrm{BG}}$. Then the expectation values of the non-constant operators at later times are given from (35) and

$$\langle a_1\rangle = z_1(t) \qquad \langle a_2\rangle = z_2(t) \qquad \langle b^1\rangle = \zeta_1(t) \qquad \langle b^2\rangle = \zeta_2(t) \tag{53}$$

where $z_1(t)=z_1S$, $z_2(t)=z_2S$, $\zeta_1(t)=\zeta_1S$, $\zeta_2(t)=\zeta_2S$, and

$$\begin{aligned}S &= \sum_{n=1}^{\infty}\sum_{m=-(n-1)}^{n-1}\sideset{}{'}\sum_{p=-(n-1-|m|)}^{n-1-|m|} L(n,m,p,t)\\ L(n,m,p,t) &= \frac{|P(n,m,p,z_1,z_2,\zeta_1,\zeta_2)|^2}{I_0(2\mu)A(n,m,p)}F(n,t)\\ F(n,t) &= \exp\left(\frac{-\mathrm{i}t(n+\frac{1}{2})}{n^2(n+1)^2}\right).\end{aligned} \tag{54}$$

The sum S appears to be intractable, but it is the asymptotic behaviour of S which interests us. By a similar 'renormalisation' method to that for the so(2, 1) case, we find for large μ,

$$S\sim\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-\tfrac{1}{2}\alpha^2)F\left(\mu+\sqrt{\frac{\mu}{2}}\,\alpha,t\right)\mathrm{d}\alpha \tag{55}$$

with F as in (54). In (55), the asymptotic behaviour of the coefficient of t in the exponential in F is given by

$$\frac{\mu+\sqrt{\mu/2}\alpha+\frac{1}{2}}{(\mu+\sqrt{\mu/2}\alpha)^2(\mu+\sqrt{\mu/2}\alpha+1)^2}=\mu^{-3}-\frac{3}{\sqrt{2}}\mu^{-7/2}\alpha+3\mu^{-4}\alpha^2-\tfrac{3}{2}\mu^{-4}+\mathrm{O}(\mu^{-9/2}). \tag{56}$$

For times t of order μ^3, i.e., times of the order of the classical period $\tau=2\pi(-2\langle H\rangle)^{-3/2}=2\pi\mu^3$, we put $t=\xi\mu^3$ with $\xi=\mathrm{O}(1)$, so that we have $S\sim\exp(-\mathrm{i}\xi)$. The expectation values of the non-constant operators therefore follow the corresponding classical trajectory: for times $t=\mathrm{o}(\mu^{7/2})$, we have

$$\begin{aligned}\langle\boldsymbol{\Gamma}(t)\pm\mathrm{i}\boldsymbol{M}(t)\rangle&\sim\langle\boldsymbol{\Gamma}(0)\pm\mathrm{i}\boldsymbol{M}(0)\rangle\exp(\pm\mathrm{i}t/\mu^3)\\ \langle\Gamma_4(t)\pm\mathrm{i}T(t)\rangle&\sim\langle\Gamma_4(0)\pm\mathrm{i}T(0)\rangle\exp(\pm\mathrm{i}t/\mu^3).\end{aligned} \tag{57}$$

From (49), (50) and (57), it can be seen that the uncertainties also maintain their minimum values over $t=\mathrm{O}(\mu^3)$, i.e., the states do not spread over times of the order of the classical period, and so can be regarded as quasiclassical for such times.

For times $t=\mathrm{O}(\mu^{7/2})$, we put $t=\sigma\mu^{7/2}$ with $\sigma=\mathrm{O}(1)$ and see that

$$S\sim\exp(-\tfrac{9}{4}\sigma^2-\mathrm{i}\sigma\sqrt{\mu})$$

so that

$$\langle\Gamma_4(t)\pm\mathrm{i}T(t)\rangle\sim\langle\Gamma_4(0)\pm\mathrm{i}T(0)\rangle\exp(-\tfrac{9}{4}\sigma^2\pm\mathrm{i}\sigma\sqrt{\mu}).$$

The expectation values therefore decay as a Gaussian function of time with a characteristic time of order $\mu^{7/2}$ (i.e. of order $\tau^{7/6}$). We now have

$$\langle M_1\rangle^2+\langle\Gamma_1\rangle^2\sim|z_1-\zeta_1|^2\exp(-\tfrac{9}{2}\sigma^2) \tag{58}$$

and so

$$(\Delta M_1)^2+(\Delta\Gamma_1)^2\sim|z_1-\zeta_1|^2(1-\exp(-\tfrac{9}{2}\sigma^2))+\lambda(\mu)+1 \tag{59}$$

with similar results for $\langle M_2\rangle^2+\langle\Gamma_2\rangle^2$, $\langle M_3\rangle^2+\langle\Gamma_3\rangle^2$, $\langle T\rangle^2+\langle\Gamma_4\rangle^2$, and the corresponding sums of squares of uncertainties. Therefore the uncertainties of the non-constant operators increase until these sums attain their maximum possible value of $\lambda(\mu)+1$. Thus the state spreads over times of order $\mu^{7/2}$, and the associated probability distribution appears to become uniformly smeared around the orbit in so(4, 2) variable space.

Just as in the so(2, 1) case, there is also unusual behaviour for times of order μ^4. If J is an integer and the time differs from $\frac{2}{3}\pi J\mu^4$ by an interval of order $\mu^{7/2}$ then $t=\frac{2}{3}\pi J\mu^4+\sigma\mu^{7/2}$ with $\sigma=\mathrm{O}(1)$. By the same sort of argument as those used in the so(2, 1) case, we get

$$S\sim\frac{(-1)^J}{\sqrt{1+\mathrm{i}4\pi J}}\exp\left(-\mathrm{i}\frac{8\pi J\mu}{3}-\mathrm{i}\sigma\sqrt{\mu}-\frac{9\sigma^2(1-\mathrm{i}4\pi J)}{4(1+16\pi^2J^2)}\right) \tag{60}$$

so that

$$\begin{aligned}\langle\boldsymbol{\Gamma}(t)\pm\mathrm{i}\boldsymbol{M}(t)\rangle&\sim\langle\boldsymbol{\Gamma}(0)\pm\mathrm{i}\boldsymbol{M}(0)\rangle\frac{(-1)^J}{\sqrt{1\mp\mathrm{i}4\pi J}}\\&\quad\times\exp\left(\pm\mathrm{i}\frac{8\pi J\mu}{3}\pm\mathrm{i}\sigma\sqrt{\mu}-\frac{9\sigma^2(1\pm\mathrm{i}4\pi J)}{4(1+16\pi^2J^2)}\right).\\ \langle\Gamma_4(t)\pm\mathrm{i}T(t)\rangle&\sim\langle\Gamma_4(0)\pm\mathrm{i}T(0)\rangle\frac{(-1)^J}{\sqrt{1\mp\mathrm{i}4\pi J}}\\&\quad\times\exp\left(\pm\mathrm{i}\frac{8\pi J\mu}{3}\pm\mathrm{i}\sigma\sqrt{\mu}-\frac{9\sigma^2(1\pm\mathrm{i}4\pi J)}{4(1+16\pi^2J^2)}\right).\end{aligned} \tag{61}$$

Therefore $\langle \boldsymbol{\Gamma} \rangle$, $\langle \boldsymbol{M} \rangle$, $\langle \Gamma_4 \rangle$ and $\langle T \rangle$ become significant at times near $\frac{2}{3}\pi J \mu^4 (= \frac{1}{3} J(\tau^4/2\pi)^{1/3})$ when the original coherence tries to 'reassert' itself. We therefore have

$$\langle M_1 \rangle^2 + \langle \Gamma_1 \rangle^2 \sim \frac{|z_1 - \zeta_1|^2}{\sqrt{1+16\pi^2 J^2}} \exp\left(-\frac{9\sigma^2}{2(1+16\pi^2 J^2)} \right) \tag{62}$$

and so

$$(\Delta M_1)^2 + (\Delta \Gamma_1)^2 \sim |z_1 - \zeta_1|^2 \left(1 - \frac{\exp(-9\sigma^2/2(1+16\pi^2 J^2))}{\sqrt{1+16\pi^2 J^2}} \right) + \lambda(\mu) + 1 \tag{63}$$

with similar results for $\langle M_2 \rangle^2 + \langle \Gamma_2 \rangle^2$, $(\Delta M_2)^2 + (\Delta \Gamma_2)^2$, etc.

These results can be summarised as follows. The states are quasiclassical for times of the order of the classical period τ, and the expectation values of the non-constant operators follow the classically predicted trajectory to within a factor of $O(\mu^{-1/2})$. The uncertainties remain constant to within a similar factor. For times of order $\tau^{7/6}$, the expectation values of the non-constant operators decay to zero and the uncertainties increase until the sum of their squares reaches the maximum possible value. The uncertainties become of the order of $\langle \Gamma_0 \rangle \sim \mu$, so that for times of this order, the states are no longer quasiclassical. Effectively, the states spread around the classical orbit. A surprising feature is that the states partially reassert their coherence, with the expectation values of non-constant operators diverging from zero, at regular intervals of $\frac{1}{3}(\tau^4/2\pi)^{1/3}$. Each of the uncertainties of the non-constant operators also diverges from its limiting value at regular intervals of $\frac{1}{3}\pi\mu^4 (= \frac{1}{6}(\tau^4/2\pi)^{1/3})$.

As in the so(2, 1) case, the quasiclassical states are all quasiperiodic. In particular, for large μ, at times which are integral multiples of $4\pi\eta^2$, where η is an integer divisible by all integers less than or equal to $\mu + \kappa\sqrt{\mu} + 1$, and greater than or equal to $\mu - \kappa\sqrt{\mu} - 1$, then

$$\langle n, m, p | \psi(t) \rangle = \langle n, m, p | \psi(0) \rangle$$

if $\mu - \kappa\sqrt{\mu} - 1 \leqslant n \leqslant \mu + \kappa\sqrt{\mu} + 1$, and so

$$\Re\{\langle \psi(t) | \psi(0) \rangle\} \geqslant 2 \operatorname{erf} \kappa - 1 + O(\mu^{-1/2}).$$

Then, with S as in (54),

$$\Re(S) \geqslant 2 \operatorname{erf} \kappa - 1 + O(\mu^{-1/2}),$$

so that the expectation values $\langle \boldsymbol{\Gamma} \rangle$, $\langle \boldsymbol{M} \rangle$, $\langle \Gamma_4 \rangle$ and $\langle T \rangle$ approach their initial values arbitrarily closely, and their uncertainties approach their initial (minimum) values arbitrarily closely. Just as in the so(2, 1) case,

$$\ln \eta = 2\kappa\sqrt{\mu} \ln \mu (1 + o(1)).$$

The time required for quasiperiodicity is typically extremely large, as discussed at the end of the last section.

5. Conclusion

We have defined quasiclassical (or quasicoherent) states for the Coulomb problem, based on Barut-Girardello coherent states of an so(4, 2) dynamical algebra, and we have determined some of their basic properties. In particular, we have seen that these states evolve in such a way under the Coulomb Hamiltonian that the degree of spreading is insignificant for times of the order of the corresponding classical period τ, though significant for times of order $\tau^{7/6}$.

We believe that these states may be of considerable interest for various applications. Yeazell and Stroud (1988) have recently observed quasiclassical states for the sodium atom. These states are localised with respect to the polar coordinates θ and ϕ but not with respect to the radial distance r. They were produced experimentally by acting on sodium atoms with a short-pulse optical excitation in the presence of a strong background radiation field. Our approach could be modified to define states of alkali atoms which act in a quasiclassical way, by taking the atom as a central core (the positively charged ion) plus a single electron, so that the atom is approximately hydrogen-like. This approximation will not be good for low energies (near the ground state) but can be expected to be very close for highly excited states. The quasiclassical states which are of major interest are therefore the ones corresponding to large quantum numbers, so that our analysis should be of relevance in this case. It may soon be possible to observe experimentally, quasiclassical near-ionisation states of hydrogen, and it would be very interesting to see if fluctuations at times of order $\tau^{4/3}$ (i.e. 'resurgence of coherence') could be observed.

Quasicoherent states of hydrogen (or alkali atoms) might well be useful in the description of interactions between radiation and atoms of this type in the near-ionisation region.

They might also be used to define a basis of states for multi-electron atoms, where quasicoherent states of the whole system could be constructed as antisymmetrised tensor products of individual Coulomb-system coherent states.

Acknowledgments

We thank A O Barut for inspiration and encouragement, and G Milburn for a helpful comment. DSM acknowledges the financial support of a Commonwealth Postgraduate Research Award.

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